Optical Gap Solitons in Nonresonant Quadratic Media

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We demonstrate an important role of the process of optical rectification in the theory of nonlinear wave propagation in quadratically nonlinear [or $\chi^{(2)}$] periodic optical media. We derive *a novel physical model* for gap solitons in $\chi^{(2)}$ nonlinear Bragg gratings.

As has been recently demonstrated, large optical nonlinearities can be generated in noncentrosymmetric media by means of the so-called *cascading effects*, due to wave mixing under the condition of nearly phase-matched second-harmonic generation and other parametric processes [1]. It has been also shown that cascaded nonlinearities can support spatial optical solitons [2] and also different types of gap solitons in periodic Bragg gratings with a quadratic [or $\chi^{(2)}$] nonlinear response [3].

Importantly, when an input electromagnetic wave $\mathbf{E}$ at frequency ω is launched into a noncentrosymmetric material, it generates also a quasi-static electric field (or *dc field*) at frequency zero. This effect is known as *optical rectification*, and it is usually described by a contribution to the medium nonlinear polarization P of the form $P_i^0 = \epsilon_0 \chi_{ijk}(0;\omega,-\omega) E_j(\omega) E_k^*(\omega)$, where $\chi_{ijk}(0;\omega,-\omega)$ is the nonlinear optical susceptibility describing optical rectification [4]. Such an induced dc field changes a refractive index via the linear electro-optic effect. As has been recently shown by Bosshard *et al.* [5], both theoretically and experimentally, the combined processes of optical rectification and the linear electro-optic effect lead to an *additional, nonresonant* contribution into an effective nonlinear refractive index of noncentrosymmetric materials due to cascading processes.

The effect of optical rectification is *usually neglected* in the theory of quadratic solitons because the equation for the dc field can be integrated explicitly, leading to a nonresonant contribution into the effective cubic nonlinearity of the nonlinear Schrödinger (NLS) equation derived by means of the asymptotic technique in the approximation of cascaded nonlinearities (see, e.g., Ref. [6]). However, for the propagation of spatio-temporal multi-dimensional optical pulses in nonresonant quadratic media, such a reduction is no longer possible and, as a result, the multi-dimensional NLS equation becomes coupled to a dc field [7], similarly to the integrable case of the Dawey-Stewartson equation [8].

In this paper we show that the physical situation is qualitatively different for periodic quadratically nonlinear optical media. We demonstrate that coupling between the forward and backward waves in one-dimensional shallow Bragg gratings with a quadratic nonlinearity is accompanied by a coupling to the induced dc field that appears within the same approximation and cannot be eliminated by integration. *This effect has been overlooked previously*, but it leads to *a novel physical model for gap solitons* in quadratic media which we introduce and analyze here.

We consider propagation of an optical pulse in a periodic medium with a quadratic $\chi^{(2)}$ nonlinear response. To derive the coupled-mode equations for the wave envelope, we start from Maxwell's equation,

$$c^2 \nabla^2 E - \frac{\partial^2}{\partial t^2}\left[\hat{\epsilon}(z, i\partial_t) + \chi^{(2)} E\right] E = 0, \qquad (1)$$

where ∇^2 stands for the Laplacian, c is the speed of light in vacuum, E is the x-element of the electric field, $\boldsymbol{E} = E(z,t)\boldsymbol{e_x}$, and the quadratic nonlinearty is represented by a tensor element $\chi^{(2)} = \boldsymbol{\chi^{(2)}_{xxx}}$. We assume that $\hat{\epsilon}(z,\omega)$ is a periodic function of z, so it can be presented in a general form as a Fourier series,

$$\hat{\epsilon}(z,\omega) = \epsilon(\omega)\left(1 + \sum_{j=1}^{\infty} \epsilon_j e^{2ikz} + \sum_{j=1}^{\infty} \epsilon_j^* e^{-2ikz}\right), \qquad (2)$$

where $d = \pi/k$ is the period of the Bragg-grating structure. Deriving the couple-mode equations below, we assume the case of a *shallow grating*, i.e. that the condition $\epsilon_j \ll 1$ holds. Additionally, we may consider a periodic modulation of the nonlinear quadratic susceptibility taking $\chi^{(2)}(z) = \chi^{(2)}(z+d)$ as a periodic function with the same period d. However, we have verified that this effect does not modify qualitatively the analysis and results presented below, so that we consider the simplest case when $\chi^{(2)}$ is constant.

For a periodic structure, the Bragg reflection leads to a strong interaction between the forward and backward waves at the Bragg wavenumber $k_B \approx k$. To derive the coupled-mode equations for the wave envelopes, we consider the asymptotic expansion for the electric field in the form,

$$\begin{aligned} E &= \left(E_+ e^{ikz} + E_- e^{-ikz}\right) e^{-i\omega t} + \text{c.c.} \\ &+ E^{(0,0)} + E^{(0,2)} e^{2ikz} + E^{(0,-2)} e^{-2ikz} \\ &+ \left(E^{(2,0)} + E^{(2,2)} e^{2ikz} + E^{(2,-2)} e^{-2ikz}\right) e^{-2i\omega t} + \text{c.c.}, \end{aligned} \qquad (3)$$

where $E_\pm = E_\pm(z,t)$ are slowly varying envelopes of the forward(+) and bakward(-) waves. The frequency ω

satisfies the dispersion relation for linear waves, $c^2k^2 = \omega^2\epsilon(\omega)$. Due to quadratic nonlinearity, the expansion (3) includes higher-order terms at the frequency 2ω and the zero-frequency term, so that the slowly varying functions $E^{(n,m)} = E^{(n,m)}(z,t)$ are defined as nonlinear amplitudes of the (n,m)-order harmonics $e^{-in\omega t}e^{imkz}$.

Introducing a small parameter ε, we assume $E_\pm \sim O(\varepsilon)$, $\partial E_\pm/\partial t \sim \partial E_\pm/\partial z \sim O(\varepsilon^3)$ and $\epsilon_j \sim O(\varepsilon^2)$. Then, substituting the expansion (2), (3) into Eq. (1), we compare the terms of the same order in front of the coefficients $e^{-in\omega t}e^{imkz}$. At the orders $(2,0)$, $(2,\pm 2)$ and $(0,\pm 2)$ we respectively obtain,

$$E^{(2,0)} = -\frac{2\chi^{(2)}}{\epsilon(2\omega)}E_+E_- \sim O(\varepsilon^2),$$

$$E^{(2,\pm 2)} = -\frac{\omega^2\chi^{(2)}}{[c^2k^2 - \omega^2\epsilon(2\omega)]}E_\pm^2 \sim O(\varepsilon^2),$$

$$E^{(0,\pm 2)} = -\frac{\chi^{(2)}}{2c^2k^2}\frac{\partial^2}{\partial t^2}\left(E_\pm^* E_\mp\right) \sim O(\varepsilon^6),$$

where we have assumed non-resonant interaction with the second harmonic, i.e. $\omega^2\epsilon(2\omega) \neq c^2k^2$.

At the orders $(1,\pm 1)$ and $(0,0)$ we obtained a system of coupled nonlinear equations,

$$i\left(\frac{\partial}{\partial t} + v_g\frac{\partial}{\partial z}\right)E_+ + \kappa E_- + \left(A|E_+|^2 + B|E_-|^2 + CE^{(0,0)}\right)E_+ = 0, \quad (4)$$

$$i\left(\frac{\partial}{\partial t} - v_g\frac{\partial}{\partial z}\right)E_- + \kappa^* E_+ + \left(B|E_+|^2 + A|E_-|^2 + CE^{(0,0)}\right)E_- = 0, \quad (5)$$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v_0^2}\frac{\partial^2}{\partial t^2}\right)E^{(0,0)} + D\frac{\partial^2}{\partial t^2}\left(|E_+|^2 + |E_-|^2\right) = 0, \quad (6)$$

where $v_g(\omega) = d\omega/dk$, $v_0 = v_g(0)$, $\kappa = \omega^2\epsilon(\omega)\epsilon_1 f^{-1}(\omega)$, $A = 2(\chi^{(2)})^2\omega^4\{f(\omega)[c^2k^2 - \omega^2\epsilon(2\omega)]\}^{-1}$, $B = -4(\chi^{(2)})^2\omega^2[f(\omega)\epsilon(2\omega)]^{-1}$, $C = 2\omega^2\chi^{(2)}f^{-1}(\omega)$, and $D = -2\chi^{(2)}/c^2$ with $f(\omega) \equiv [\omega^2\epsilon(\omega)]'$. If we keep the transverse coordinates (x,y), Eq. (6) should also include the transverse Laplacian in the same order. System (4)-(6) describes the interaction between the forward and backward waves coupled to a dc wave induced via the rectification effect. Importantly, the order of the dc wave $E^{(0,0)}$ is ε^2, so that the dc field itself is of a higher order in comparison with the forward and backward scattering waves. However, the dc field is coupled to the fields E_+ and E_- in the main order. Importantly, if we assume much stronger dc field, e.g. of order of $O(1)$, the system is decoupled and the dc wave satisfies an independent equation. Including the optical Kerr effect, we obtain the same equations as Eqs. (4), (5) with the modified constants A and B.

In the case of a single wave propagating in a homogeneous medium, the induced dc field is explicitly given by the host wave [6]. The similar result is valid for the case of a deep grating described by the modulations of the Bloch waves, but not for a shallow grating we discuss here. If we assume the scaling $\partial E_\pm/\partial z \sim O(\varepsilon^2)$ (as usually done in the analysis of higher-dimensional systems such as the Dawey-Stewartson equation) and $E^{(0,0)}$ of order ε^4, then the coupling between the dc wave and host-wave scan be neglected. However, for *isotropic scaling* as presented here, the effect of the dc wave is directly included into Eqs. (4) and (5). Interaction between the dc field and fundamental harmonics has been also discussed in Ref. [6], however in that analysis, the dc field appears as a cascading effect and its velocity is almost the same as the phase velocity. We notice that in our case, the dc field is essentially excited by quadratic nonlinearity and no assumption is required for the velocity.

We are looking for spatially localized solutions of Eqs. (4)-(6) for *bright gap solitons* in the form,

$$E_+ = \Delta^{-1/2}f(\zeta)e^{i[\theta_1(\zeta)-\Omega t+g/2]},$$
$$E_- = \Delta^{1/2}f(\zeta)e^{i[\theta_2(\zeta)-\Omega t-g/2]}. \quad (7)$$

where $\zeta = z - Vt$; the functions $f(\zeta)$ and $\theta_{1,2}(\zeta)$, and the parameters Ω, V, Δ are assumed to be real. The parameter g is the argument of the coupling parameter κ, i.e. $\kappa = |\kappa|e^{ig}$. Substituting the ansatz (7) into Eq. (6), we obtain

$$E^{(0,0)}(\zeta) = -\frac{v_0^2V^2D}{(v_0^2 - V^2)}\left(\Delta + \frac{1}{\Delta}\right)f^2(\zeta),$$

and therefore the contribution of the dc field should vanish at $V = 0$.

From Eqs. (4) and (5), we set the parameter Δ as $\sqrt{(v_g - V)/(v_g + V)}$, and then obtain a system of coupled equations for f, $\theta_- \equiv \theta_1 - \theta_2$ and $\theta_+ \equiv \theta_1 + \theta_2$,

$$\frac{df}{d\zeta} = \mu f\sin\theta_-, \quad (8)$$

$$\frac{d\theta_-}{d\zeta} + \nu - 2\mu\cos\theta_- + \delta f^2 = 0, \quad (9)$$

$$\frac{d\theta_+}{d\zeta} + \frac{V\nu}{v_g} + \eta f^2 = 0, \quad (10)$$

where $\mu = |\kappa|/(v_g^2 - V^2)^{1/2}$, $\nu = -(2v_g\Omega)/(v_g^2 - V^2)$, and

$$\delta = -2(v_g^2 - V^2)^{-1/2}\left(\frac{(v_g^2 + V^2)}{(v_g^2 - V^2)}\tilde{A} + \tilde{B}\right),$$
$$\eta = -4(v_g^2 - V^2)^{-3/2}v_gV\tilde{A},$$
$$\tilde{A} = A - \frac{V^2v_0^2CD}{(v_0^2 - V^2)}, \qquad \tilde{B} = B - \frac{V^2v_0^2CD}{(v_0^2 - V^2)}.$$

We notice that in the problem under consideration, we should assume $|V| < v_g$ and $|\nu| < 2\mu$.

Similar to the analysis presented in Ref. [9], from Eqs. (8)-(10) we obtain a closed differential equation for the function θ_- in the form of the double sine-Gordon (DSG) equation. The DSG equation can be integrated, and its localized solutions are two types of *kinks* and *anti-kinks* [10]. Using the relevant solutions, we can then find

$$f(\zeta) = \left\{ \frac{\mp(4\mu/\delta)[1-(\nu/2\mu)^2]}{\cosh(\zeta\sqrt{4\mu^2-\nu^2}) \mp (\nu/2\mu)} \right\}^{1/2},$$

where the signs $\pm$ stand for the cases $\delta > 0$ and $\delta < 0$, respectively. Functions θ_1 and θ_2 are then obtained as

$$\theta_- = \theta_1 - \theta_2$$
$$= -2\tan^{-1}\left\{ \sqrt{\frac{2\mu-\nu}{2\mu+\nu}}\ \tanh^{\mp 1}\left(\frac{\sqrt{4\mu^2-\nu^2}}{2}\zeta\right)^{\mp 1} \right\}, \quad (11)$$

$$\theta_+ = \theta_1 + \theta_2 = -\frac{\nu V}{v_q}\zeta$$
$$\pm\frac{4\eta v_g}{\delta}\tan^{-1}\left[\sqrt{\frac{2\mu\pm\nu}{2\mu\mp\nu}}\tanh\left(\frac{\sqrt{4\mu^2-\nu^2}}{2}\zeta\right)\right] + \mathcal{C}_\pm, \quad (12)$$

where $\mathcal{C}_\pm$ are integration constants. In order to obtain solutions for gap solitons, we restrict the possible angle variable by the domain, $0 \le \theta_- \le 2\pi$. The solution obtained from Eq. (11) describes a two-parameter family of gap solitons, spatially localized waves in the Bragg gratings, which are similar to the gap solitons of the conventional couple-mode theory. Actually, by renormalizing the variables as $A \to \pm\sigma$, $B \to \pm 1$, $C \to 0$, $|\kappa| \to 1$, $v_g \to 1$, $V \to v$, $\Omega \to \pm\sqrt{1-v^2}\cos Q$, $\mathcal{C}_\pm \to (1/2 \pm 1/2)\pi \pm (4\sigma v\alpha^2)/(1-v^2)\pi + 2\phi$, we can demonstrate that the solution is essentially the same as that earlier obtained in Ref. [11]. However, the effect of the dc wave is included in the parameters δ and η.

In the case $|\nu| > 2\mu$, spatially localized solution of Eqs. (8), (9) do not exist. Instead, the kinks of the DSG equation for θ_- give solutions for *dark gap solitons* (see also [12]), localized waves on nonvanishing backgrounds,

$$f(\zeta) = \sqrt{\frac{2\mu}{|\delta|}\left(\frac{|\nu|}{2\mu} - 1\right)}\frac{\sqrt{|\nu|/2\mu}\cosh(\sigma\zeta) \pm 1}{\sqrt{(|\nu|/2\mu)\cosh^2(\sigma\zeta) - 1}},$$

where

$$\sigma = 4\mu\sqrt{\frac{|\nu|}{2\mu} - 1}.$$

Upper and lower signs correspond to two types of such solitons, with the maximum intensity *large* or *smaller* than the background intensity.

The effective renormalisation of the coefficients due to the induced dc field seems extremely important for the soliton stability. Indeed, when $v_0 < v_g$ the coefficients have a singularity provided $V \to v_0$, changing the character of the dependence of the soliton parameters and the system conserved quantities on V. The recent stability analysis of the conventional gap solitons [13] revealed the existence of two types of instabilities, *oscillatory* and *translational*. The most important, translational instability appears for large V, so that the induced dc field is expected to have a strong effect on the soliton stability. In fact, we anticipate that all gap solitons for $v_0 < V < v_g$ may become unstable, similar to the solitary waves in hydrogen-bonded molecular systems [14]. It is also worth to mention that in the limit $V \to v_0$ when the coefficients grow, the transverse effect in Eq. (6) become important and should be included to compensate the singularity.

To demonstrate that the issue of the soliton stability becomes nontrivial for this model, we present the system invariants. Similar to some other models, we are not able to present Eqs. (4)-(6) in a Hamiltonian form directly, and therefore we introduce an auxiliary function ϕ through the relation, $\alpha\partial\phi/\partial z = E^{(0,0)} - v_0^2 D(|E_+| + |E_-|^2)$, where $\alpha^2 = v_0^2 D/C$. Then, we define the second canonical variable as $\psi = v_0^2\partial\phi/\partial t$ and show that Eqs. (4)-(6) can be written as a Hamiltonian system

$$\frac{\partial\phi}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\phi}, \quad \frac{\partial E_\pm}{\partial t} = i\frac{\delta H}{\delta E_\pm^*},$$

with the following Hamiltonian,

$$H = \int_{-\infty}^{+\infty} dz \left\{ \frac{v_0^2}{2}\psi^2 - \phi\frac{\partial^2\phi}{\partial z^2} + \kappa E_+^* E_- + \kappa^* E_+ E_-^* \right.$$
$$+\frac{iv_g}{2}\left(E_+^*\frac{\partial E_+}{\partial z} - E_+\frac{\partial E_+^*}{\partial z} - E_-^*\frac{\partial E_-}{\partial z} + E_-\frac{\partial E_-^*}{\partial z}\right)$$
$$+\frac{\bar{A}}{2}(|E_+|^4 + |E_-|^4) + \bar{B}|E_- E_+|^2$$
$$\left. +\alpha C\frac{\partial\phi}{\partial z}\left(|E_+|^2 + |E_-|^2\right) \right\}, \quad (13)$$

where $\bar{A} = A + v_0^2 CD$ and $\bar{B} = B + v_0^2 CD$. Other integrals of motion of the system (4)-(6) are the field momentum,

$$P = \int_{-\infty}^{+\infty} dz \left\{ \left(E_+\frac{\partial E_+^*}{\partial z} + E_-\frac{\partial E_-^*}{\partial z}\right) - 2\frac{\partial\phi}{\partial z}\psi \right\},$$

the total number of the forward and backward waves, and an independently conserved number of the dc waves,

$$N = \int_{-\infty}^{+\infty} dz(|E_+|^2 + |E_-|^2), \quad N_0 = \int_{-\infty}^{+\infty} dz\psi.$$

Therefore, in a sharp contrast to the conventional theory of gap solitons, the model (4)-(6) possesses one additional

integral of motion, it has no analogy with other soliton-bearing nonintegrable models where the soliton stability has been investigated so far.

In conclusion, we have demonstrated that in periodic optical media with a quadratic nonlinear response, Bragg-grating-induced coupling between the forward and backward propagating modes leads always to an induced dc field which plays an important role for nonlinear pulse propagation in periodic $\chi^{(2)}$ media. We have derived a novel model of the coupled-mode theory for optical gap solitons in quadratically nonlinear Bragg gratings, and have found the analytical solutions for moving bright and dark gap solitons. Analysis of the soliton stability for the model (4)-(6) will be presented elsewhere.

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